Gauge Unification within the Dual Standard Model

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We calculate the gauge couplings in the dual standard model. These values are consistent with an associated GeV mass scale, and predict the weak mixing angle to be $\sin^2\theta_w(M_Z) \sim 0.22$.

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The standard model fermions have an intricate representation structure under the colour, weak isospin and hypercharge symmetry groups. The five basic multiplets (replicated in three generations) divide into leptons and quarks corresponding to trivial and fundamental representations of the colour symmetry. These leptons and quarks subdivide further corresponding to the trivial or fundamental representations of weak isospin, with this division coinciding with left and right parity eigenstates.

Currently, the *only* explanation for such a structure is the dual standard model of Vachaspati [1]. Here the standard model fermions are associated with monopoles originating from the symmetry breaking of a unified $SU(5)$ gauge theory to the standard model gauge symmetry. The representation structure, and hence interaction, of these monopoles is in exact agreement with the spectrum of fermions in the standard model.

In addition, other properties such as the spin [2], and the number of generations can be consistently included within this framework [3]. Its structure may also be related to confinement within QCD [4,5].

In this letter we calculate the colour, weak isospin and hypercharge gauge coupling constants of these $SU(5)$ monopoles. Essentially our calculation compares the gauge transformation properties of the monopoles with the associated fermions. This uniquely specifies the colour, weak isospin and hypercharge gauge coupling constants in terms of the unified $SU(5)$ coupling. We find these gauge couplings to be consistent with the experimental values.

We begin by summarising some of the main features of the dual standard model [1,5]. The model originates with a breaking of $SU(5)$ gauge symmetry

$$\begin{aligned} SU(5) &\to S(U(3)\times U(2)) \\ &= [SU(3)_\mathrm{C}\times SU(2)_\mathrm{I}\times U(1)_\mathrm{Y}]/\mathbf{Z}_6 \end{aligned} \tag{1}$$

and has a monopole spectrum corresponding to the homotopy classes

$$\begin{aligned} \pi_2\left(\frac{SU(5)}{S(U(3)\times U(2))}\right) &\cong \pi_1(S(U(3)\times U(2))) \\ &= \mathbf{Z}_6\times\mathbf{Z}. \end{aligned} \tag{2}$$

Here $\mathbf{Z}$ defines the degree of the homotopy class, whilst

$$\mathbf{Z}_6 = \mathbf{Z}_3\times\mathbf{Z}_2 = \{e^{2i\pi/3}, e^{-2i\pi/3}, 1\}\times\{-1,1\} \tag{3}$$

represents second homotopy classes of same degree.

The monopoles spectrum is built up from bound states of embedded $SU(2)\to U(1)$ fundamental monopoles,

$$\begin{array}{ccc} SU(5) & \to & S(U(3)\times U(2)) \\ \cup & & \cup \\ SU(2) & \to & U(1), \end{array} \tag{4}$$

and correspond to the $(e^{2i\pi/3},-1)$ homotopy class of $\mathbf{Z}_6$. Gardner and Harvey [6] show that these fundamental monopoles combine to form stable bound states for a natural range of model parameters. Labelling the bound states by their asymptotic magnetic fields

$$B^k \sim \frac{\hat{r}^k}{r^2}Q, \tag{5}$$

defines an associated magnetic charge

$$Q = \frac{1}{g_u}\left(q_\mathrm{C}X_\mathrm{C} + q_\mathrm{I}X_\mathrm{I} + q_\mathrm{Y}X_\mathrm{Y}\right), \tag{6}$$

where $X_\mathrm{C}, X_\mathrm{I}$ and X_Y are suitably normalised elements of the Lie algebras $su(3), su(2)$ and $u(1)$. The coefficient $1/g_u$ relates to the unified $SU(5)$ gauge coupling g_u.

The magnetic charges are determined by associating them with the corresponding homotopy classes in Eq. (2). They define a subgroup

$$U(1)_Q = \exp(\mathbf{R}Q) \subset S(U(3)\times U(2)), \tag{7}$$

normalised by

$$\exp(2\pi g_u Q) = 1. \tag{8}$$

This subgroup represents a typical element of the associated $\mathbf{Z}_6$ homotopy class of the monopole. Using generators

$$X_\mathrm{C} = i\ \mathrm{diag}(-\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{2}{3}, 0, 0), \tag{9}$$

$$X_\mathrm{I} = i\ \mathrm{diag}(0,0,0,1,-1), \tag{10}$$

$$X_\mathrm{Y} = i\ \mathrm{diag}(1,1,1,-\tfrac{3}{3},-\tfrac{3}{2}) \tag{11}$$

leads to the following pattern for the monopole spectrum:

	q_C	q_I	q_Y	d_C	d_I	d_Y
$(e^{2i\pi/3},-1)$	1	1/2	1/3	3	2	1
$(e^{-2i\pi/3},1)$	-1	0	2/3	3	0	1
$(1,-1)$	0	-1/2	1	0	2	1
$(e^{2i\pi/3},1)$	1	0	4/3	3	0	1
$(e^{-2i\pi/3},-1)$	-	-	-	-	-	-
$(1,1)$	0	0	2	0	0	1

It should be noted that we have chosen a slightly different normalisation from [1,5]. This is to agree with the standard particle physics charge normalisations.

Degeneracies $d_{\rm C}$, $d_{\rm I}$ and $d_{\rm Y}$ of the monopole embeddings corresponding to the same homotopy class have also been included. These arise from the degeneracy of suitable generators

$$X_{\rm C}^r = i\ {\rm diag}(+\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 0, 0), \tag{12}$$
$$X_{\rm C}^g = i\ {\rm diag}(-\tfrac{1}{3}, +\tfrac{2}{3}, -\tfrac{1}{3}, 0, 0), \tag{13}$$
$$X_{\rm C}^b = i\ {\rm diag}(-\tfrac{1}{3}, -\tfrac{1}{3}, +\tfrac{2}{3}, 0, 0) \tag{14}$$

for $X_{\rm C}$ and

$$X_{\rm I}^{\pm} = \pm X_{\rm I} \tag{15}$$

for $X_{\rm I}$. This indicates that the monopoles form representations of $SU(3)_{\rm C}$, $SU(2)_{\rm I}$ and $U(1)_{\rm Y}$ with the corresponding dimension. Namely the fundamental representations.

The above arguments strongly imply that the long range interactions of these monopoles is associated with that of a particle with gauge interactions specified by the fundamental representations of the colour, weak isospin and hypercharge symmetry groups. This particle has the corresponding charges $q_{\rm C}$, $q_{\rm I}$ and $q_{\rm Y}$, and its current $J^\mu_{\rm mon}$ couples to the gauge fields as

$$[g_{\rm C} q_{\rm C} A^\mu_{\rm C} + g_{\rm I} q_{\rm I} A^\mu_{\rm I} + g_{\rm Y} q_{\rm Y} A^\mu_{\rm Y}]\, J^\mu_{\rm mon}, \tag{16}$$

with $g_{\rm C}$, $g_{\rm I}$ and $g_{\rm Y}$ representing the respective gauge couplings. Such a spectrum of charges and interactions is completely in accord with the spectrum of fermions in the standard model, with the identification:

$$\begin{array}{rcl}
(e^{2i\pi/3}, -1) & \leftrightarrow & \begin{pmatrix} u \\ d \end{pmatrix}_L \\
(e^{-2i\pi/3}, 1) & \leftrightarrow & \bar{d}_L \\
(1, -1) & \leftrightarrow & \begin{pmatrix} \bar{\nu} \\ \bar{e} \end{pmatrix}_R \\
(e^{2i\pi/3}, 1) & \leftrightarrow & u_R \\
(1, 1) & \leftrightarrow & \bar{e}_L
\end{array} \tag{17}$$

The corresponding fermionic anti-particles are associated with the anti-monopoles.

The main point of this work is to show that as well as predicting the spectrum and properties of fermions in the standard model, the dual standard model also predicts the corresponding colour, weak isospin and hypercharge gauge couplings. We shall determine these from the gauge transformation properties of the monopoles, presented below. A more detailed treatment will be given elsewhere [7].

One may see simply that three different gauge couplings should arise in Eq. (16) by considering the normalisation of the monopole charge generators in Eqs. (9, 10, 11). These generators $X_{\rm C}$, $X_{\rm I}$ and $X_{\rm Y}$ are normalised to the topology of $S(U(3)\times U(2))$. However the gauge fields of $SU(5)$ theory are normalised differently. In the minimal coupling the components of the gauge fields are written

$$D^\mu = \partial^\mu + g_u A^\mu_a \hat{X}_a, \tag{18}$$

with the $SU(5)$-basis $\{\hat{X}_a\}$ orthonormal with respect to the inner product

$${\rm tr}(\hat{X}_a \hat{X}_b) = \tfrac{1}{2}\delta_{ab}. \tag{19}$$

The difference between these normalisations will produce overall scales associated with the gauge couplings.

To be more specific, recall that the fundamental monopoles are embedded $SU(2)\to U(1)$ monopoles, described by Eq. (4), with magnetic fields B^k corresponding to the embedding. Rigid (or global) monopole gauge transformations that respect $B^k \in su(3)_{\rm C} \oplus su(2)_I \oplus u(1)_{\rm Y}$ transform

$$B^k \mapsto {\rm Ad}(h) B^k \tag{20}$$

under the adjoint action of $h \in S(U(3)\times U(2))$. Correspondingly the $su(2)$ embedding transforms under

$$su(2) \mapsto {\rm Ad}(h) su(2), \tag{21}$$

so that Q transforms appropriately.

Considering a gauge transformation of the embedded monopole, the generators must be normalised to the $SU(5)$ gauge interaction of Eqs. (18, 19). Typical normalised colour, weak isospin and hypercharge generators are

$$\hat{X}_{\rm C} = \frac{\sqrt{3}}{2} X_{\rm C}, \quad \hat{X}_{\rm I} = \frac{1}{2} X_{\rm I}, \quad \hat{X}_{\rm Y} = \frac{1}{\sqrt{15}} X_{\rm Y}. \tag{22}$$

From which the corresponding rigid gauge transformations of the monopole take

$$B^k \mapsto {\rm Ad}[\exp(g_u(\hat{X}_{\rm C}\theta_1 + \hat{X}_{\rm I}\theta_2 + \hat{X}_{\rm Y}\theta_3))] B^k. \tag{23}$$

Those taking $B^k \mapsto B^k$ are thus

$$\theta_1 = \frac{2}{\sqrt{3}} \frac{2\pi}{g_u} n_1, \quad \theta_2 = 2\frac{2\pi}{g_u} n_2, \quad \theta_3 = \sqrt{15}\frac{2\pi}{g_u} n_3, \tag{24}$$

with $n_1, n_2, n_3 \in \mathbf{N}$.

Now we shall consider the analagous rigid gauge transformations on a fermion f in the fundamental representation of $SU(3)_{\rm C} \times SU(2)_{\rm I} \times U(1)_{\rm Y}/\mathbf{Z}_6$. We associate with X_C, X_I, X_Y the generators

$$i\,\lambda_8 = i\,{\rm diag}(1,1,-2), \tag{25}$$
$$i\,\sigma_3 = i\,{\rm diag}(1,-1), \tag{26}$$
$$i\,\mathbf{1}_2 = i\,{\rm diag}(1,1). \tag{27}$$

Their action upon the fermions must be are normalised to the standard particle physics normalisations

$$\tfrac{1}{12}\mathrm{tr}(\lambda_8^2) = \tfrac{1}{4}\mathrm{tr}(\sigma_3^2) = \tfrac{1}{4}\mathrm{tr}(\mathbf{1}_2^2) = \tfrac{1}{2}. \quad (28)$$

Then the corresponding rigid gauge transformations are

$$f \mapsto e^{ig'\bar{\theta}_3/2}\exp(\tfrac{1}{\sqrt{12}} i g_s \lambda_8 \bar{\theta}_1)\ f\ \exp(\tfrac{1}{2} i g \sigma_3 \bar{\theta}_2), \quad (29)$$

with g_s, g and g' the colour, weak isospin and hypercharge gauge couplings. Those rigid gauge transformation that take $f \mapsto f$ are thus

$$\bar{\theta}_1 = \sqrt{12}\frac{2\pi}{g_s}n_1,\ \ \bar{\theta}_2 = 2\frac{2\pi}{g}n_2,\ \ \bar{\theta}_3 = 2\frac{2\pi}{g'}n_3, \quad (30)$$

with $n_1, n_2, n_3 \in \mathbf{N}$.

Equating monopole and fermion gauge transformation identifies $\theta_i = \bar{\theta}_i$ in Eqs. (24) and (30). This gives:

$$g_s = 4g_u, \quad g = g_u, \quad g' = \frac{2}{\sqrt{15}}g_u, \quad (31)$$

which predicts the following ratios:

$$\frac{g_s}{g} = 4,\ \ \frac{g'}{g} = \frac{2}{\sqrt{15}}. \quad (32)$$

Such values represent a specific prediction of the dual standard model and are completely characteristic of it.

These predictions are compared to the running gauge couplings through the following plot. The strong coupling is taken from a three loop calculation normalised to $g_s(M_Z) = 1.213$. The hypercharge and weak isospin are taken from one loop expressions normalised to $g(M_Z) = 0.661$ and $g'(M_Z) = 0.354$.

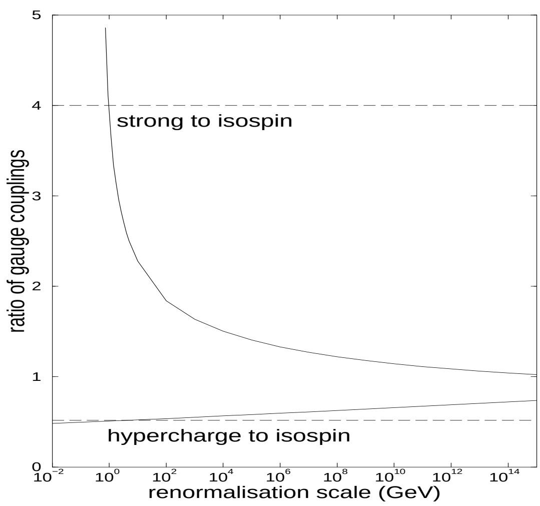

FIG. 1. g_s/g and g'/g plotted against renormalisation scale μ. Dual standard model predicted values are also included.

We shall make a couple of comments about the running of the gauge couplings in the standard model. Firstly, around $10^{15} - 10^{18}$GeV, when $g_s/g \sim 1$ then also $g'/g \sim \sqrt{3/5}$, as required for grand unification *. Secondly, g'/g runs below the Z-mass from the running of the fine structure constant α. Its form may be estimated by the following relation [8]

$$M_W = \frac{A_0}{\sin\theta_w(1-\Delta r)^{1/2}}, \quad A_0 = (\pi\alpha/\sqrt{2}G_F)^{1/2}, \ (33)$$

with Δr representing the radiative corrections. Its component from the running of α is $\Delta r_0(\mu) = (1 - \alpha/\alpha(\mu))$.

The conclusion from fig. (1) is that the dual standard model is associated with a mass scale of about one GeV. At that scale the running couplings take the values of *both* of our theoretical predictions in Eq. (32). However, there is some uncertainty in this value due to the extrapolation of the running couplings to one GeV. Taking this into account implies that a more accurate mass scale would be somewhere between about three quarters of a GeV and two GeV.

To illustrate the accuracy of the fit in fig. (1) we shall calculate a prediction for $\sin^2\theta_w(M_Z)$ using only the running of the strong coupling and Eq. (32). Firstly observe that $g_s/g = 4$ is satisfied at around a GeV. Then Eq. (32) implies that $\sin^2\theta_w = 4/19$ at the same scale. Using Eq. (33), we predict

$$\sin^2\theta_w(M_Z) \sim \frac{\alpha}{\alpha(M_Z)}\sin^2\theta_w(0) \sim 0.22. \quad (34)$$

The experimental value is $\sin^2\theta_w(M_Z) = 0.2230 \pm 0.0004$.

We now discuss the interpretation and implication of the above results. We give two possible interpretations.

In the conventional picture the standard model unifies into a grand unified theory at some large energy scale, say around $10^{15} - 10^{18}$GeV. In this framework we would interpret the standard model fermion spectrum and interactions as being dual to the grand unified monopoles because of some higher symmetry, perhaps arising from string theory or some other method of unification. By the above arguments this would naturally give rise to a mass scale of around a GeV for the standard model fermions, and would represent a possible explanation for the hierarchy between the unified and the fermion mass scales. In addition the above arguments support the $SU(5)$ group as a candidate for grand unification.

Alternatively, there is a more radical proposition: the observed fermions are actually monopoles and they are formed at an energy scale of around a GeV, where the fundamental gauge symmetries unify.

We now briefly discuss some issues related to this proposal.

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*the $\sqrt{3/5}$ factor arises through the normalisation of the hypercharge generator.

In collider type experiments at momentum scales in excess of a few GeV one would not expect to see symmetry restoration. Monopoles are interpreted as composite coherent bound states of scalar and gauge particles. As a composite object they still retain their identity at energies in excess of their formation scale. For instance protons remain hadronic objects far in excess of $\Lambda_{\rm QCD} \sim 300$MeV. To observe the restoration of symmetry one would have to form an analogous object to a chiral condensate, with a temperature of the unification scale. Then the monopoles would decompose into their constituent fields.

The gauge interaction of the monopoles is also as observed for the gauge interactions of the fundamental fermions. They transform equivalently under the same symmetries and, providing these symmetries are gauged, they will interact in the same fashion.

A problem with many low energy theories of unification is the presence of proton decay. In the context of the dual standard model it is not clear that unification at such a scale would cause proton decay via the massive lepto-quark gauge bosons. These gauge bosons are in the dual sector to the monopoles, and would therefore appear magnetically charged. As such they would interact with the fermions in an entirely different way from the usual lepton or baryon number violating interactions.

Taking all this into consideration, still the unification scale does seem rather low. Considering the monopoles as classical objects then their composite nature should become apparent at momentum scales above the unification scale. This is not seen. Precisions test of the standard model limit compositeness to scales above a TeV.

This may not be a problem though. One should really consider a quantum theory of monopoles. In such a quantum theory a classical argument about the internal structure of the monopoles may not apply. Currently the quantisation of the monopole degrees of freedom is an unsolved problem. However results from duality seem to imply that a theory of monopoles is dual to a particle theory. If a quantised theory of monopoles does result in a particle theory then the dual standard model could well be a totally consistent explanation for the form and structure of the standard model.

In conclusion we have examined the gauge couplings in the dual standard model. This model represents a theoretically well motivated explanation of the spectrum and interaction of the standard model fermions. We have shown that it predicts $g_s/g = 4$ and $g'/g = 2/\sqrt{15}$, values that are consistent with a renormalisation scale around a GeV. In the context of the dual standard model the interpretation of this result is ambiguous: it could either represent a natural explanation for the fermion mass-unification scale hierarchy, or it could indicate the presence of new physics at low energy scales.

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